

Stochastic Analysis on Path Space over Time-Inhomogeneous Manifolds with Boundary

Li-Juan Cheng *

(School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,
Beijing 100875, The People's Republic of China)
E-mail: chenglj@mail.bnu.edu.cn(L.J. Cheng)

Abstract

Let $L_t := \Delta_t + Z_t$ for a $C^{1,1}$ -vector field Z on a differential manifold M with possible boundary ∂M , where Δ_t is the Laplacian induced by a time dependent metric g_t differentiable in $t \in [0, T_c)$. We first introduce the damp gradient operator, defined on the path space with reference measure \mathbb{P} , the law of the (reflecting) diffusion process generated by L_t on the base manifold; then establish the integration by parts formula for underlying directional derivatives and prove the log-Sobolev inequality for the associated Dirichlet form, which is further applied to the free path spaces; and finally, establish numbers of transportation-cost inequalities associated to the uniform distance, which are equivalent to the curvature lower bound and the convexity of the boundary.

Keywords: Metric flow, log-Sobolev inequality, integration by parts formula, path space over manifolds with boundary, reflecting L_t -diffusion process

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1 Introduction

Let M be a d -dimensional differential manifold with possible boundary ∂M equipped with a complete Riemannian metric $(g_t)_{t \in [0, T_c)}$, which is C^1 in t . For simplicity, we take the notations: for $X, Y \in TM$,

$$\begin{aligned}\mathrm{Ric}_t^Z(X, Y) &:= \mathrm{Ric}_t(X, Y) - \langle \nabla_X^t Z_t, Y \rangle_t, \quad \mathcal{G}_t(X, X) := \partial_t g_t(X, X) \\ \mathcal{R}_t^Z(X, Y) &:= \mathrm{Ric}_t^Z(X, Y) - \frac{1}{2} \mathcal{G}_t(X, Y),\end{aligned}$$

where Ric_t is the Ricci curvature tensor with respect to g_t , $(Z_t)_{t \in [0, T_c)}$ is a C^1 -family of vector fields, and $\langle \cdot, \cdot \rangle_t := g_t(\cdot, \cdot)$. Define the second fundamental form of the boundary by

$$\mathbb{I}_t(X, Y) = -\langle \nabla_X^t N_t, Y \rangle_t, \quad X, Y \in T\partial M,$$

where N_t is the inward unit normal vector field of the boundary associated with g_t ; $T\partial M$ is the tangent space of ∂M . Consider the elliptic operator $L_t := \Delta_t + Z_t$. Let X_t be the reflecting inhomogeneous diffusion process generated by L_t (called reflecting L_t -diffusion process). Assume

*Correspondence should be addressed to Li-Juan Cheng (E-mail: chenglj@mail.bnu.edu.cn)

that X_t is non-explosive before T_c . In this case, for any $0 \leq S < T < T_c$, the distribution $\Pi^{S,T}$ of $X_{[S,T]} := \{X_t : t \in [S, T]\}$ is a probability measure on the path space

$$W^{S,T} := C([S, T]; M),$$

when $S = 0$, we write $W^T := W^{0,T}$ and $\Pi^T := \Pi^{0,T}$ for simplicity. For each point $x \in M$, let $X_{[0,T]}^x = \{X_t^x : 0 \leq t \leq T < T_c\}$ and $W_x^T = \{\gamma \in W^T : \gamma_0 = x\}$. It has been well-known that there is a strong connection between the behavior of the distribution of the M -valued Brownian motion associated with metric g and the geometry of their underlying space. This paper is devoted to further the study of this relation over a inhomogeneous manifold. Note that although our discussions base on the manifold with boundary, the results are also new for the manifold without boundary.

When the metric is independent of t , the theory about stochastic analysis on the path space over a complete Riemannian manifold has been well developed since Driver [9] proved the quasi-invariance theorem for the Brownian motion. Then, integration by parts formula for the associated gradient operator, induced by the quasi-invariant flow, was established, which leads to the study of the functional inequalities with respect to the corresponding Dirichlet form (see e.g. [16, 12]). For the case with boundary, see [18, 25] for the corresponding results on manifold with boundary; see [3, 5, 11] for an intertwining formula for the differential of Itô development map.

A probabilistic approach to these problem was initiated by Abandon et al, who constructed g_t -Brownian motion on time-inhomogeneous space in [2]. Recently, Chen [6] gives the intertwining formula and log-Sobolev inequality for usual Dirichlet form (without damp) on the path space over time-inhomogeneous manifold. The main purpose of this paper is to prove the integration by parts formula and further establish the log-Sobolev inequality w.r.t. the associated Dirichlet form, defined by the damped gradient operator. Note that from technical point of view, our method relies on [18, 25].

Our second purpose is to discuss the Talagrand type transportation-cost inequalities on the path space with respect to the uniform distance on time-inhomogeneous space. In 1996, Talagrand [20] found that the L^2 -Wasserstein distance to the standard Gaussian measure can be dominated by the square root of twice relative entropy on $M = \mathbb{R}^d$ with constant metric. This inequality has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of diffusion processes with respect to several different distances (i.e. cost functions). See [14] for the details on the Wiener space with the Cameron-Martin distance; see [21, 8] on the path space of diffusions with the L^2 -distance; see [22] on the Riemannian path space with intrinsic distance induced by the Malliavin gradient operator, and [13, 27, 24] on the path space of diffusions with the uniform distance.

The rest parts of the paper are organized as follows. In the following two sections, we construct the Hsu's multiplicative functional and then define the corresponding damped gradient operator, which satisfies an integration by parts formula induced by intrinsic quasi-invariant flows. In Section 4, the log-Sobolev inequality for the associated Dirichlet form is established

and extends to the free path space. In Section 5, some transportation-cost inequalities are presented to be equivalent to the curvature condition and the convexity of the boundary, and parts of these are extended to non-convex case in final section.

2 (Reflecting) L_t -diffusion process and multiplicative functional

Let $\mathcal{F}(M)$ be the frame bundle over M and $\mathcal{O}_t(M)$ be the orthonormal frame bundle over M with respect to g_t . Let $\mathbf{p} : \mathcal{F}(M) \rightarrow M$ be the projection from $\mathcal{F}(M)$ onto M . Let $\{e_\alpha\}_{\alpha=1}^d$ be the canonical orthonormal basis of \mathbb{R}^d . For any $u \in \mathcal{F}(M)$, let $H_i^t(u)$ be the ∇^t horizontal lift of ue_i and $\{V_{\alpha,\beta}(u)\}_{\alpha,\beta=1}^d$ be the canonical basis of vertical fields over $\mathcal{F}(M)$, defined by $V_{\alpha,\beta}(u) = Tl_u(\exp(E_{\alpha,\beta}))$, where $E_{\alpha,\beta}$ is the canonical basis of $\mathcal{M}_d(\mathbb{R})$, the $d \times d$ matrix space over \mathbb{R} , and $l_u : Gl_d(\mathbb{R}) \rightarrow \mathcal{F}(M)$ is the left multiplication from the general linear group to $\mathcal{F}(M)$, i.e. $l_u \exp(E_{\alpha,\beta}) = u \exp(E_{\alpha,\beta})$.

Let $B_t := (B_t^1, B_t^2, \dots, B_t^d)$ be a \mathbb{R}^d -valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Assume the elliptic generator L_t is a C^1 functional of time with associated metric g_t :

$$L_t = \Delta_t + Z_t$$

where Z_t is a $C^{1,1}$ vector field on M . As in the time-homogeneous case, to construct the L_t -diffusion process, we first construct the corresponding horizontal diffusion process generated by $L_{\mathcal{O}_t(M)} := \Delta_{\mathcal{O}_t(M)} + H_{Z_t}^t$ by solving the Stratonovich stochastic diffusion equation (SDE):

$$\begin{cases} du_t = \sqrt{2} \sum_{i=1}^d H_i^t(u_t) \circ dB_t^i + H_{Z_t}^t(u_t) dt - \frac{1}{2} \sum_{i,j} \partial_t g_t(u_t e_i, u_t e_j) V_{i,j}(u_t) dt + H_{N_t}^t(u_t) dl_t, \\ u_0 \in \mathcal{O}_0(M), \end{cases}$$

where $\Delta_{\mathcal{O}_t(M)}$ is the horizontal Laplace operator on $\mathcal{O}_t(M)$; $H_{Z_t}^t$ and $H_{N_t}^t$ are the ∇^t horizontal lift of Z_t and N_t respectively; l_t is an increasing process supported on $\{t \geq 0 : X_t := \mathbf{p}u_t \in \partial M\}$. By a similar discussion as in [2, Proposition 1.2], we see that the last term promises $u_t \in \mathcal{O}_t(M)$. Since $(H_{Z_t}^t)_{t \in [0, T_c]}$ is a $C^{1,1}$ -family vector field, it is well-known that (see e.g. [17]) the equation has a unique solution up to the life time $\zeta := \lim_{n \rightarrow \infty} \zeta_n$, and

$$\zeta_n := \inf\{t \in [0, T_c) : \rho_t(\mathbf{p}u_0, \mathbf{p}u_t) \geq n\}, \quad n \geq 1, \quad \inf \emptyset = T_c,$$

where $\rho_t(x, y)$ is the distance between x and y associated with g_t . Let $X_t = \mathbf{p}u_t$. It is easy to see that X_t solves the equation

$$dX_t = \sqrt{2}u_t \circ dB_t + Z_t(X_t)dt + N_t(X_t)dl_t, \quad X_0 = x := \mathbf{p}u$$

up to the life time ζ . By the Itô formula, for any $f \in C_0^{1,2}([0, T_c) \times M)$ with $N_t f_t := N_t f_t|_{\partial M} = 0$,

$$f(t, X_t) - f(0, x) - \int_0^t (\partial_s + L_s) f(s, X_s) ds = \sqrt{2} \int_0^t \langle u_s^{-1} \nabla^s f(s, \cdot)(X_s), dB_s \rangle$$

is a martingale up to the life time ζ . Here and what follows, we denote the inner product on \mathbb{R}^d by $\langle \cdot, \cdot \rangle$, and write $f(t, \cdot) = f_t$ for simplicity. So, we call X_t the reflecting diffusion process generated by L_t . When $Z_t \equiv 0$, then $\tilde{X}_t := X_{t/2}$ is generated by $\frac{1}{2}\Delta_t$ and is called the reflecting g_t -Brownian motion on M . In what follows, we assume the process is non-explosive.

2.1 Multiplicative functional

To construct the desired continuous multiplicative functional, we need the following assumption.

(A) There exist two constants $K, \sigma \in C([0, T_c])$ such that

$$\mathcal{R}_t^Z \geq K(t), \mathbb{I}_t \geq \sigma(t) \text{ and } \mathbb{E} e^{\lambda \int_0^t \sigma^-(s) dl_s^x} < \infty$$

holds for $\lambda > 0$, $t \in [0, T_c]$, and $x \in M$, where $\sigma^- = 0 \vee (-\sigma)$.

To introduce Hsu's discontinuous multiplicative functional, we need the lift operators of \mathcal{R}_t^Z , \mathcal{G}_t , \mathbb{I}_t . For any $u \in \mathcal{O}_t(M)$, let $\mathcal{R}_u^Z(t)(a, b) = \mathcal{R}_t^Z(ua, ub)$ and $\mathcal{G}_u(t)(a, b) = \mathcal{G}_t(ua, ub)$, $a, b \in \mathbb{R}^d$. Let $\mathbf{p}_\partial : TM \rightarrow T\partial M$ be the orthogonal projection at points on $(\partial M, g_t)$. For any $u \in \mathcal{O}_t(M)$ with $\mathbf{p}u \in \partial M$, let

$$\mathbb{I}_u(t)(a, b) = \mathbb{I}_t(\mathbf{p}_\partial ua, \mathbf{p}_\partial ub), a, b \in \mathbb{R}^d.$$

For $u \in \partial \mathcal{O}_t(M)$, the boundary of $\mathcal{O}_t(M)$, let

$$P_u(t)(a, b) = \langle ua, N_t \rangle_t \langle ub, N_t \rangle_t, a, b \in \mathbb{R}^d.$$

For any $\varepsilon > 0$ and $r \geq 0$, let $Q_{r,t}^{x,\varepsilon}$ solve the following SDE on $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$dQ_{r,t}^{x,\varepsilon} = -Q_{r,t}^{x,\varepsilon} \left\{ \mathcal{R}_{u_t^x}^Z(t) dt + (\varepsilon^{-1} P_{u_t^x}(t) + \mathbb{I}_{u_t^x}(t)) dl_t^x \right\}. \quad (2.1)$$

When the metric is independent of t and M is compact, letting $\varepsilon \downarrow 0$, the process $Q_{r,t}^{x,\varepsilon}$ converges in L^2 to an adapted right-continuous process $Q_{r,t}^x$ with left limit, such that $Q_{r,t}^x P_{u_t^x}(t) = 0$ if $X_t^x \in \partial M$ (see [18, Theorem 3.4]). Here, we follow Wang [26, Theorem 4.1.1], and introduce a slightly different but simpler construction of the multiplicative functional by solving a random integral equation on $\mathbb{R}^d \otimes \mathbb{R}^d$.

Theorem 2.1. *Assume (A), then*

(1) *for any $x \in M$, $0 \leq r \leq t < T_c$ and $u_0^x \in \mathcal{O}_0(M)$, the equation*

$$Q_{r,t}^x = \left(I - \int_r^t Q_{r,s}^x \mathcal{R}_{u_s^x}^Z(s) ds - \int_s^t Q_{r,s}^x \mathbb{I}_{u_s^x}(s) dl_s^x \right) (I - 1_{\{X_t^x \in \partial M\}} P_{u_t^x}(t))$$

has a unique solution;

(2) *for any $0 \leq r \leq t < T_c$, $\|Q_{r,t}^x\| \leq e^{-\int_r^t K(s) ds - \int_r^t \sigma(s) dl_s^x}$ a.s., where $\|\cdot\|$ is the operator norm for $d \times d$ -matrices;*

(3) *for any $0 \leq r \leq s \leq t < T_c$, $Q_{r,t}^x = Q_{r,s}^x Q_{s,t}^x$ a.s..*

Proof. We only consider the existence of the solution up to a arbitrarily given time $T \in (r, T_c)$, since the uniqueness is obvious. In the following, we drop the superscript x for simplicity. By the assumption (A) and (2.1), we have

$$\|Q_{r,t}^\varepsilon\|^2 \leq 1 - 2 \int_r^t \|Q_{r,s}^\varepsilon\|^2 K(s) ds - 2 \int_r^t \|Q_{r,s}^\varepsilon\|^2 \sigma(s) dl_s - \frac{2}{\varepsilon} \int_s^t \|Q_{r,s}^\varepsilon P_{u_s}(s)\|^2 dl_s, \quad t > r.$$

Therefore, we obtain $\|Q_{r,t}^\varepsilon\|^2 \leq e^{-2 \int_r^t K(s) ds - 2 \int_r^t \sigma(s) dl_s}$, $t \geq r$ and

$$\int_r^T \|Q_{r,s}^\varepsilon P_{u_s}(s)\|^2 dl_s \leq \frac{\varepsilon}{2} \left[1 + 2 \left(\int_r^T K^-(s) ds + \int_r^T \sigma^-(s) dl_s \right) e^{2 \int_r^T K^-(s) ds + \int_r^T \sigma^-(s) dl_s} \right]. \quad (2.2)$$

Combining this with (A), we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \int_r^T \|Q_{r,s}^\varepsilon P_{u_s}(s)\|^2 dl_s = 0 \quad (2.3)$$

and

$$\sup_{\varepsilon \in (0,1)} \mathbb{E} \int_r^T \|Q_{r,t}^\varepsilon\|^2 (dt + dl_t) < \infty.$$

Because of the latter, we may select a sequence $\varepsilon_n \downarrow 0$ and an adapted process $\overline{Q}_{r,\cdot} \in L^2(\Omega \times [r, T]) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d; \mathbb{P} \times (dt + dl_t)$, such that for any $g \in L^2(\Omega \times [r, T]) \rightarrow \mathbb{R}^d; \mathbb{P} \times (dt + dl_t)$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_r^T Q_{r,t}^{\varepsilon_n} g_t (dt + dl_t) = \mathbb{E} \int_r^T \overline{Q}_{r,t} g_t (dt + dl_t).$$

The following discussion is almost the same as the case with constant metric, see [25] for details. \square

The following result is a direct conclusion of Theorem 2.1.

Proposition 2.2. *Assume (A). For any \mathbb{R}^d -valued continuous semi-martingale h_t with*

$$1_{\{X_t^x \in \partial M\}} P_{u_t}(t) h_t = 0,$$

$$dQ_{r,t}^x h_t = \overline{Q}_{r,t}^x dh_t - Q_{r,t}^x \mathcal{R}_{u_t^x}^Z(t) h_t dt - Q_{r,t}^x \mathbb{I}_{u_t^x}(t) h_t dl_t^x, \quad t \geq r,$$

where

$$\overline{Q}_{r,t}^x = \left(I - \int_r^t Q_{r,s}^x \mathcal{R}_{u_s^x}^Z(s) ds - \int_r^t Q_{r,s}^x \mathbb{I}_{u_s^x}(s) dl_s^x \right).$$

Proof. As $1_{\{X_t^x \in \partial M\}} P_{u_t}(t) h_t = 0$, and by Theorem 2.1, we have

$$Q_{r,t}^x h_t = \left(I - \int_r^t Q_{r,s}^x \mathcal{R}_{u_s^x}^Z(s) ds + \frac{1}{2} \int_r^t Q_{r,s}^x \mathcal{G}_{u_s^x}(s) ds - \int_r^t Q_{r,s}^x \mathbb{I}_{u_s^x}(s) dl_s^x \right) h_t = \overline{Q}_{r,t}^x h_t.$$

Then the proof is completed by using Itô formula. \square

Recall that $\{P_{r,t}\}_{0 \leq r \leq t < T_c}$ is the Neumann semigroup generated by L_t . The following is a consequence of Proposition 2.2, which is the derivative formula of the diffusion semigroup, known as Bismut-Elworthy Li formula (see e.g. [3, 10]).

Corollary 2.3. Assume (A). Let $f \in C_b^\infty(M)$. Then for any $0 \leq r < t < T_c$,

$$[r, t] \ni s \rightarrow Q_{r,s}^x (u_s^x)^{-1} \nabla^s P_{s,t} f(X_s^x)$$

is a martingale. Consequently,

$$(u_r^x)^{-1} \nabla^r P_{r,t} f(X_r^x) = \mathbb{E}(Q_{r,t}^x (u_t^x)^{-1} \nabla^t f(X_t^x) | \mathcal{F}_r), \quad (2.4)$$

and for any adapted \mathbb{R}_+ -valued process ξ satisfying $\xi(r) = 0, \xi(t) = 1$, and $\mathbb{E}(\int_r^t \xi'(s)^2 ds)^\alpha < \infty$ for $\alpha > 1/2$, there holds

$$(u_r^x)^{-1} \nabla^r P_{r,t} f(X_r^x) = \frac{1}{\sqrt{2}} \mathbb{E} \left(f(X_t^x) \int_r^t \xi'(s) (Q_{r,s}^x)^* dB_s | \mathcal{F}_r \right).$$

Proof. The proof is essentially due to [25, Corollary 3.4]. Without losing generality, we assume $r = 0$ and drop the superscript x for simplicity.

We first prove that $Q_s h_s$ is a martingale. Let $h_s = (u_s)^{-1} \nabla^s f(X_s)$. Since $\nabla^s P_{s,t} f$ is vertical to N_s on ∂M , $1_{\{X_s \in \partial M\}} P_u(s) h_s = 0$. Then we have

$$dQ_s h_s = \overline{Q}_s dh_s - Q_s \mathcal{R}_u^Z(s) h_s ds - Q_s \mathbb{I}_u(s) h_s dl_s.$$

Let $F(u, s) := u^{-1} \nabla^s P_{s,t} f(\mathbf{p}u)$, $u \in \mathcal{O}_s(M)$, then

$$\frac{d}{ds} F(u, s) = -u^{-1} \nabla^s P_{s,t} f(\mathbf{p}u) = -L_{\mathcal{O}_s(M)} F(\cdot, s)(u) + (\mathcal{R}_u^Z(s) + \frac{1}{2} \mathcal{G}_u(s)) F(u, s), s \in [0, t].$$

On the other hand, noting that

$$du_t = \sqrt{2} \sum_{i=1}^d H_i^t \circ dB_t^i + H_{Z_t}^t(u_t) dt - \frac{1}{2} \sum_{\alpha, \beta=1}^d \mathcal{G}_{\alpha, \beta}(t, u_t) V_{\alpha, \beta}(u_t) dt + H_{N_t}^t(u_t) dl_t.$$

By the Itô formula, we have

$$\begin{aligned} dF(u_s, t_0) &= dM_s + L_{\mathcal{O}_s M} F(\cdot, t_0)(u_s) ds + H_{N_s}^s F(\cdot, t_0)(u_s) dl_s \\ &\quad - \frac{1}{2} \sum_{\alpha, \beta} \mathcal{G}_{\alpha, \beta}(s, u_s) V_{\alpha, \beta}(u_s) F(\cdot, t_0)(u_s) ds. \end{aligned} \quad (2.5)$$

where

$$dM_s := \sqrt{2} \sum_{i=1}^d H_i^s F(\cdot, t_0)(u_s) dB_s^i.$$

Therefore,

$$dh_s = dM_s + \mathcal{R}_u^Z(s) h_s ds + H_{N_s}^s F(\cdot, s)(u_s) dl_s.$$

Since $1_{\{X_s \in \partial M\}} Q_s P_{u_s}(s) = 0$, combining this with (2.5), we obtain

$$dQ_s h_s = Q_s dM_s + Q_s (I - P_{u_s}(s)) \{ H_{N_s}^s F(\cdot, s)(u_s) - \mathbb{I}_u(s) F(u_s, s) \} dl_s.$$

Noting that for any $e \in \mathbb{R}^d$, it follows from that when $X_s \in \partial M$,

$$\begin{aligned} \langle (I - P_{u_s}(s)) H_{N_s}^s F(\cdot, s)(u_s), e \rangle &= \text{Hess}_{P_{s,t} f}^s(N_s, \mathbf{p}_{\partial} u_s e) = \mathbb{I}_s(\nabla^s P_{s,t} f(X_s), \mathbf{p}_{\partial} u_s e) \\ &= \mathbb{I}_u(s)(F(u, s), e) = \langle \mathbb{I}_u(s) F(u, s), e \rangle, \end{aligned}$$

we conclude that

$$(1 - P_u(s))\{H_{N_s}^s F(\cdot, s)(u_s) - \mathbb{I}_u(s)F(u_s, s)\}dl_s = 0.$$

Therefore, $Q_s h_s$ is a local martingale. By **(A)**, it is then a martingale according to [7, Theorem 4.9].

The following step is similar as shown in step (b) in the proof of [25, Lemma 3.3]. We skip it here. □

3 Damped Gradient, quasi-invariant flows and integration by Parts

When the metric is independent of t , the Malliavin derivative can be realized by quasi-invariant flows for diffusion on manifolds (see e.g. [17, 25]). In this section, by using the multiplicative functional constructed in §2.1, we first introduce the damped gradient operator as in [11], then introduce quasi-invariant flows induced by SDEs with refraction, and finally link them by establishing an integration by parts formula.

3.1 Damped Gradient operator and quasi-invariant flows

We shall use multiplicative functionals $\{Q_{r,t}^x : 0 \leq r \leq t < T_c\}$ to define the damped gradient operator for functionals of X^x .

Let

$$\mathcal{F}C_0^\infty = \{W^T \ni \gamma \rightarrow f(\gamma_{t_1}, \gamma_{t_2}, \dots, \gamma_{t_n}) : n \geq 1, 0 < t_1 < t_2 < \dots < t_n \leq T, f \in C_0^\infty(M^n)\}$$

be the class of smooth cylindrical functions on W^T . Let

$$\mathbf{H}_0 := \left\{ h \in C([0, T]; \mathbb{R}^d) : h(0) = 0, \|h\|_{\mathbf{H}_0} := \int_0^T |\dot{h}(s)|^2 ds < \infty \right\}$$

be the Cameron-Martin space on the flat path space. For any $F \in \mathcal{F}C_0^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \gamma_{t_2}, \dots, \gamma_{t_n})$, define the damped gradient $D^0 F(X_{[0,T]}^x)$ as an \mathbf{H}_0 -valued random variable by setting $(D^0 F(X_{[0,T]}^x))(0) = 0$ and

$$\frac{d}{dt}(D^0 F(X_{[0,T]}^x))(t) = \sum_{i=1}^n 1_{\{t < t_i\}} Q_{t,t_i}^x (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x), \quad t \in [0, T].$$

where $\nabla_i^{t_i}$ denotes the gradient operator w.r.t. the i -th component associated with g_{t_i} . Then, for any \mathbf{H}_0 -valued random variable h , let

$$D_h^0 F(X_{[0,T]}^x) = \langle D^0 F(X_{[0,T]}^x), h \rangle_{\mathbf{H}_0} = \sum_{i=1}^n \int_0^{t_i} \langle (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x), (Q_{t,t_i}^x)^* h'(t) \rangle dt,$$

We would like to indicate that the formulation of $D_h^0 F$ is consistent with [11] for the case with constant metric. Note that compared with usual gradient operator, it contains the multiplicative functional, which affects the log-Sobolev constant. This operator links $D_h^0 F$ to the directional

derivative induced by a quasi-invariant flow. We now turn to investigating this relation. The main idea essentially due to [18], where quasi-invariant flows are constructed for constant manifold M with boundary ∂M . Let $\tilde{\mathbf{H}}_0$ be the set of all adapted elements in $L^2(\Omega \rightarrow \mathbf{H}_0; \mathbb{P})$; i.e.

$$\tilde{\mathbf{H}}_0 = \{h \in L^2(\Omega \rightarrow \mathbf{H}_0; \mathbb{P}) : h(t) \text{ is a } \mathcal{F}_t\text{-measurable, } t \in [0, T]\}.$$

Then, $\tilde{\mathbf{H}}_0$ is a Hilbert space with inner product

$$\langle h, \tilde{h} \rangle_{\tilde{\mathbf{H}}_0} := \mathbb{E} \langle h, \tilde{h} \rangle_{\mathbf{H}_0} = \mathbb{E} \int_0^T \langle h'(t), \tilde{h}'(t) \rangle dt, h, \tilde{h} \in \tilde{\mathbf{H}}_0.$$

For $h \in \tilde{\mathbf{H}}_0$ and $\varepsilon > 0$, let $X_t^{\varepsilon, h}$ solve the SDE

$$dX_t^{\varepsilon, h} = \sqrt{2}u_t^{\varepsilon, h} \circ dB_t + Z_t(X_t^{\varepsilon, h})dt + \varepsilon \sqrt{2}u_t^{\varepsilon, h} h'(t)dt + N_t(X_t^{\varepsilon, h})dl_t^{\varepsilon, h}, \quad (3.1)$$

where $l_t^{\varepsilon, h}$ and $u_t^{\varepsilon, h}$ are, respectively, the local time on ∂M and the horizontal lift on $\mathcal{O}_t(M)$ for $X_t^{\varepsilon, h}$. The detailed construction of $X_t^{\varepsilon, h}$ is similar as in Section 2. To see that $\{X_{[0, T]}^{\varepsilon, h}\}_{\varepsilon \geq 0}$ has the flow property also in our setting, let

$$\Theta : W_0 := \{\omega \in C([0, T]; \mathbb{R}^d) : \omega_0 = 0\} \rightarrow W^T$$

be measurable such that $X = \Theta(B)$, $B \in W_0$. For any $\varepsilon > 0$ and a function $\Phi : W_0 \rightarrow W^T$, let $(\theta_\varepsilon^h \Phi)(\omega) = \Phi(\omega + \varepsilon h)$. Then $X_{[0, T]}^{\varepsilon, h} = (\theta_\varepsilon^h \Theta)(B)$, $\varepsilon > 0$. Hence,

$$X_{[0, T]}^{\varepsilon_1 + \varepsilon_2, h} = \theta_{\varepsilon_1}^h X_{[0, T]}^{\varepsilon_2, h}, \quad \varepsilon_1, \varepsilon_2 \geq 0.$$

Moreover, let us explain that the flow is quasi-invariant, i.e. for each $\varepsilon \geq 0$, the distribution of $X_{[0, T]}^{\varepsilon, h}$ is absolutely continuous w.r.t. that of $X_{[0, T]}^x$. Let

$$R^{\varepsilon, h} = \exp \left[\varepsilon \int_0^T \langle h'(t), dB_t \rangle - \frac{\varepsilon^2}{2} \int_0^T |h'(t)|^2 dt \right].$$

By the Girsanov theorem

$$B_t^{\varepsilon, h} := B_t - \varepsilon h(t)$$

is the d -dimensional Brownian motion under the probability $R^{\varepsilon, h} \mathbb{P}$. Thus, the distribution of $X_{[0, T]}^x$ under $R^{\varepsilon, h}$ coincides with that of $X_{[0, T]}^{\varepsilon, h}$ under \mathbb{P} . Therefore, the map $X_{[0, T]}^x \rightarrow X_{[0, T]}^{\varepsilon, h}$ is quasi-invariant. The quasi-invariant property leads us to prove the following property.

Proposition 3.1. *Let $x \in M$ and $F \in \mathcal{F}C^\infty$. Then*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \frac{F(X_{[0, T]}^{\varepsilon, h}) - F(X_{[0, T]}^x)}{\varepsilon} = \mathbb{E} \left\{ F(X_{[0, T]}^x) \int_0^T \langle h'(t), dB_t \rangle \right\}$$

holds for $h \in \tilde{\mathbf{H}}_{0, b}$, the set of all elements in $\tilde{\mathbf{H}}_0$ with bounded $\|h\|_{\tilde{\mathbf{H}}_0}$.

Proof. As we have explained that $B_t^{\varepsilon, h} = B_t - \varepsilon h(t)$ is a d -dimensional Brownian motion under $R^{\varepsilon, h} \mathbb{P}$. By the weak uniqueness of (3.1), we conclude that the distribution of X^x under $R^{\varepsilon, h} \mathbb{P}$ coincide with that of $X^{\varepsilon, h}$ under \mathbb{P} . In particular, $\mathbb{E} F(X_{[0, T]}^{\varepsilon, h}) = \mathbb{E}[R^{\varepsilon, h} F(X_{[0, T]}^x)]$. Thus, the assertion follows from $\frac{dR^{\varepsilon, h}}{d\varepsilon}|_{\varepsilon=0}$ and the dominated convergence theorem since $\{R^{\varepsilon, h}\}_{\varepsilon \in [0, 1]}$ is uniformly integrable for $h \in \tilde{\mathbf{H}}_{0, b}$. \square

3.2 Integration by parts formula

In this section, an integration by parts formula for $D_h^0 F$ is established and applied to clarifying the link between this formula and the derivative induced by the flow $\{X_{[0,T]}^{\varepsilon,h}\}_{\varepsilon \geq 0}$. The main result of this subsection is presented as follows.

Theorem 3.2. *Assume (A). For any $x \in M$ and $F \in \mathcal{F}C_0^\infty$,*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \frac{F(X_{[0,T]}^{\varepsilon,h}) - F(X_{[0,T]}^x)}{\varepsilon} = \mathbb{E}\{D_h^0 F\}(X_{[0,T]}^x) = \mathbb{E} \left\{ F(X_{[0,T]}^x) \int_0^T \langle h'(t), dB_t \rangle \right\} \quad (3.2)$$

holds for all $h \in \tilde{\mathbf{H}}_{0,b}$.

Proof. By the Proposition 3.1, it is sufficient for us to prove the second equality. The proof is similar to the constant metric case (see [25, Theorem 2.1]) due to the following Lemmas 3.3 and 3.5 and the Markov property, . Note that the second equality holds for all $h \in \tilde{\mathbf{H}}_0$. \square

The following lemma gives the gradient formula for special cylinder functions.

Lemma 3.3. *For any $n \geq 1$, $0 < t_1 < \dots < t_n \leq T$, and $f \in C^\infty(M^n)$,*

$$(u_{t_1}^x)^{-1} \nabla_1^{t_1} \mathbb{E}\{f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x) | \mathcal{F}_{t_1}\} = \sum_{i=1}^n \mathbb{E}\{Q_{t_1, t_i}^x (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x) | \mathcal{F}_{t_1}\}$$

holds for all $x \in M$ and $u_0^x \in \mathcal{O}_0(M)$, where $\nabla_i^{t_i}$ denotes the g_{t_i} -gradient w.r.t. the i -th component.

Proof. It is obvious that the assertion is true for $n = 1$. By (2.4), we have

$$(u_{t_1}^x)^{-1} \nabla_1^{t_1} \mathbb{E}(f(X_{t_2}) | \mathcal{F}_{t_1}) = \mathbb{E}(Q_{t_1, t_2}^x (u_{t_2}^x)^{-1} \nabla_2^{t_2} f(X_{t_2}) | \mathcal{F}_{t_1}).$$

which plus the case of $n = 1$, we prove the result for $n = 2$. Assume that it holds for $n = k$, $k \geq 2$. It remains to prove the case for $n = k + 1$. To this end, by Markov property, set

$$g(X_{t_1}^x, X_{t_2}^x) = \mathbb{E}f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_{k+1}}^x | \mathcal{F}_{t_2})$$

By the assumption for $n = k$, we have

$$\begin{aligned} & (u_{t_1}^x)^{-1} \nabla_1^{t_1} \mathbb{E}\{g(X_{t_1}, X_{t_2}) | \mathcal{F}_{t_1}\} \\ &= \mathbb{E}\{(u_{t_1}^x)^{-1} \nabla_1^{t_1} g(X_{t_1}^x, X_{t_2}^x) | \mathcal{F}_{t_1}\} + \mathbb{E}\{Q_{t_1, t_2}^x (u_{t_2}^x)^{-1} \nabla_2^{t_2} g(X_{t_1}, X_{t_2}) | \mathcal{F}_{t_1}\} \end{aligned} \quad (3.3)$$

for $x \in M$ and $u_0 \in \mathcal{O}_0(M)$. Fix the value of $X_{t_1}^x = x_0$, by the assumption for $n = k$, we have

$$(u_{t_2}^x)^{-1} \nabla_2^{t_2} \mathbb{E}(f(x_0, X_{t_2}^x, \dots, X_{t_{k+1}}^x) | \mathcal{F}_{t_2}) = \sum_{i=2}^{k+1} \mathbb{E}\{Q_{t_2, t_i}^x (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(x_0, X_{t_2}^x, \dots, X_{t_{k+1}}^x) | \mathcal{F}_{t_2}\}.$$

Combining this with (3.3), we have

$$\begin{aligned} & (u_{t_1}^x)^{-1} \nabla_1^{t_1} \mathbb{E}\left\{f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_{k+1}}^x) \middle| \mathcal{F}_{t_1}\right\} = (u_{t_1}^x)^{-1} \nabla_1^{t_1} \mathbb{E}\{g(X_{t_1}^x, X_{t_2}^x) | \mathcal{F}_{t_1}\} \\ &= \sum_{i=1}^{k+1} \mathbb{E}\left\{Q_{t_1, t_i}^x (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_{k+1}}^x) | \mathcal{F}_{t_1}\right\}. \end{aligned} \quad (3.4)$$

\square

Remark 3.4. Especially, choosing $t_1 = 0$, we arrive at

$$(u_0^x)^{-1} \nabla^0 \mathbb{E} \{f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x)\} = \sum_{i=1}^n \mathbb{E} \{Q_{t_i}^x (u_{t_i}^x)^{-1} \nabla_i^{t_i} f(X_{t_1}^x, X_{t_2}^x, \dots, X_{t_n}^x)\}. \quad (3.5)$$

The following result is a direct consequence of (2.4) and the Itô formula for $f(X_t^x)$, i.e.

$$f(X_t^x) = f(x) + \sqrt{2} \int_0^t \langle (u_s^x)^{-1} \nabla^s P_{s,t} f(X_s^x), dB_s \rangle.$$

Lemma 3.5. For any $n \geq 1$, $0 < t_1 < t_2 < \dots < t_n \leq T$, and $f \in C^\infty(M^n)$,

$$\mathbb{E} \left\{ f(X_t^x) \int_0^t \langle h'_s, dB_s \rangle \right\} = \mathbb{E} \int_0^t \langle (u_s^x)^{-1} \nabla^s f(X_s^x), (Q_{s,t}^x)^* h_s \rangle ds, h \in \tilde{\mathbf{H}}_0, t \in [0, T]$$

holds for all $x \in M$ and $u_0^x \in \mathcal{O}_0(M)$.

4 The Log-Sobolev Inequality

When the metric is independent of t , log-Sobolev inequalities on W_x^T were established independently by Hsu [18] and by Aida and Elworthy [1]. In this section, we first consider the path space with a fixed initiated point, then move to the free path space following an idea of [11], where the (non-damped) gradient operator is studied on the free path space over the constant manifolds without boundary.

4.1 Log-Sobolev inequality on W_x^T

Let Π_x^T be the distribution of $X_{[0,T]}^x$. Let

$$\mathcal{E}^x(F, G) = \mathbb{E} \left\{ \langle D^0 F, D^0 G \rangle_{\mathbf{H}_0} (X_{[0,T]}^x) \right\}, F, G \in \mathcal{F}C_0^\infty.$$

Since both $D^0 F$ and $D^0 G$ are functionals of X , $(\mathcal{E}^x, \mathcal{F}C_0^\infty)$ is a positive bilinear form on $L^2(W_x^T; \Pi_x^T)$. It is standard that the integration by parts formula (3.2) implies the closability of the form, see Lemma 4.1. We shall use $(\mathcal{E}^x, \mathcal{D}(\mathcal{E}^x))$ to denote the closure of $(\mathcal{E}^x, \mathcal{F}C_0^\infty)$. Moreover, (3.2) also implies the Clark-Ocône type martingale representation formula, see Lemma 4.2, which leads to the standard Gross log-Sobolev inequality (see e.g. [15]).

Lemma 4.1. Assume **(A)**. $(\mathcal{E}^x, \mathcal{F}C_0^\infty)$ is closable in $L^2(W_x^T; \Pi_x^T)$.

Proof. By the integration by part formula, the discussion is standard (see [25, Lemma 4.1]), we omit it here. \square

The following result gives us the Clark-Ocône type martingale representation formula for $F(X_{[0,T]})$. Following the proof of [25, Lemma 4.2] for the case with constant metric, we have

Lemma 4.2. (Clark-Haussman-Ocône Formula) Assume **(A)**. For any $F \in \mathcal{F}C_0^\infty$, let $\tilde{D}^0 F(X_{[0,T]}^x)$ be the projection of $D^0 F(X_{[0,T]}^x)$ on $\tilde{\mathbf{H}}_0$, i.e.

$$\frac{d}{dt} (\tilde{D}^0 F(X_{[0,T]}^x))(t) = \mathbb{E} \left(\frac{d}{dt} (D^0 F(X_{[0,T]}^x)) | \mathcal{F}_t \right), t \in [0, T], (\tilde{D}^0 F(X_{[0,T]}^x))(0) = 0.$$

Then

$$F(X_{[0,T]}^x) = \mathbb{E}F(X_{[0,T]}^x) + \int_0^T \left\langle \frac{d}{dt}(\tilde{D}^0 F(X_{[0,T]}^x))(t), dB_t \right\rangle.$$

It is standard that the martingale representation in Lemma 4.2 implies the following log-Sobolev inequality. Since the parameter T and the information of Ricci curvature and the second fundamental form have been properly contained in the Dirichlet form \mathcal{E} , the resulting log-Sobolev constant is independent of T , K and σ . Moreover, it is well-known that the constant 2 in the inequality is sharp constant for compact manifolds with constant metric.

Theorem 4.3. *Assume (A). For any $T > 0$ and $x \in M$, $(\mathcal{E}^x, \mathcal{D}(\mathcal{E}^x))$ satisfies the following log-Sobolev inequality,*

$$\Pi_x^T(F^2 \log F^2) \leq 2\mathcal{E}^x(F, F), \quad F \in \mathcal{D}(\mathcal{E}^x), \quad \Pi_x^T(F^2) = 1.$$

Proof. Due to lemma 4.1, it sufficient to prove the inequality for $F \in \mathcal{F}C_0^\infty$. Let

$$m_t := \mathbb{E} \left(F(X_{[0,T]}^x)^2 | \mathcal{F}_t \right), \quad t \in [0, T].$$

By the Itô formula,

$$dm_t \log m_t = (1 + \log m_t)dm_t + \frac{|\frac{d}{dt}(\tilde{D}^0 F(X_{[0,T]}^x))(t)|^2}{2m_t} dt.$$

Therefore,

$$\begin{aligned} \Pi_x^T(F^2 \log F^2) &= \mathbb{E}_x m_T \log m_T = \int_0^T \frac{2\mathbb{E} \left(F(X_{[0,T]}^x) \frac{d}{dt}(\tilde{D}^0 F(X_{[0,T]}^x))(t) \mid \mathcal{F}_t \right)^2}{\mathbb{E} \left(F(X_{[0,T]}^x)^2 | \mathcal{F}_t \right)} dt \\ &\leq 2 \int_0^T \mathbb{E} \left| \frac{d}{dt}(\tilde{D}^0 F(X_{[0,T]}^x))(t) \right|^2 dt \\ &= 2\mathbb{E} \|D^0 F(X_{[0,T]}^x)\|_{\mathbf{H}_0}^2 \\ &= 2\mathcal{E}^x(F, F). \end{aligned}$$

□

Note that on manifolds without boundary equipped with time-depending metric, the log-Sobolev inequality with respect to the Dirichlet form induced by usual gradient derivative is recently established in [6]. And the log-Sobolev constant is $2 \exp\{\sup_{t \in [0, T]} |\mathcal{R}_t^Z|\}$.

4.2 Application to free path spaces

Let Π_μ^T be the distribution of the (reflecting) diffusion process generated by $L_t := \nabla^t + Z_t$ with initial distribution μ and time-interval $[0, T]$. Due to the freedom of the initial point, it is natural for us to make use of the following Cameron-Martin space:

$$\mathbf{H} = \left\{ h \in C([0, T]; \mathbb{R}^d) : \int_0^T |h'(t)|^2 dt < \infty \right\}.$$

Then \mathbf{H} is a Hilbert space under the inner product

$$\langle h_1, h_2 \rangle_{\mathbf{H}} = \langle h_1(0), h_2(0) \rangle + \int_0^T \langle h'_1(t), h'_2(t) \rangle dt.$$

To defined the damped gradient operator on the free path space, let

$$\overline{\Omega} = M \times \Omega, \quad \overline{\mathcal{F}}_t = \mathcal{B}(M) \times \mathcal{F}_t, \quad \text{and} \quad \overline{\mathbb{P}} = \mu \times \mathbb{P}.$$

Let $X_0(x, \omega) = x$ for $(x, \omega) \in M \times \Omega$. Then, under the filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}})$, $X_t(x, \omega) := X_t^x(\omega)$ is the (reflecting) diffusion process generated by L_t starting from x , and $u_t(x, \omega) := u_t^x(\omega)$ is its horizontal lift. Moreover, let $Q_{r,t}(x, \omega) = Q_{r,t}^x(\omega)$ for $0 \leq r \leq t$, and write $Q_t^x(\omega) := Q_{0,t}(x, \omega)$ for simplicity. Now, for any $F \in FC_0^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_n})$, let

$$DF(X) = D^0F(X) + \sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \dots, X_{t_n}), \quad (4.1)$$

where $D^0F(X) := \sum_{i=1}^n \int_0^{t_i} Q_{t,t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, X_{t_2}, \dots, X_{t_n}) dt$ is the damped gradient on the path space with fixed initial point. Obviously, $DF(X) \in L^2(\overline{\Omega} \rightarrow \mathbf{H}; \overline{\mathbb{P}})$. Define the Dirichlet form by

$$\mathcal{E}^\mu(F, G) = \mathbb{E}_{\overline{\mathbb{P}}} \langle DF, DG \rangle_{\mathbf{H}}, \quad F, G \in \mathcal{FC}_0^\infty.$$

We aim to prove that $(\mathcal{E}^\mu, FC_0^\infty)$ is closable in $L^2(W^T; \Pi_\mu^T)$ and then establish the log-Sobolev inequality for its closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$. To prove the closability, we need the following two lemmas modified from [11]. Let $\mathcal{H}_0(M)$ be the class of all smooth vector fields on M with compact support.

Let div_μ^0 be the divergence operator w.r.t. μ , which is the minus adjoint of ∇^0 in $L^2(\mu)$; that is, for any smooth vector field U ,

$$\int_M (Uf) d\mu = - \int_M f (\text{div}_\mu^0 U) d\mu, \quad f \in C_0^1(M). \quad (4.2)$$

Lemma 4.4. *Assume (A). For any $F \in \mathcal{FC}_0^\infty$, $U \in \mathcal{H}_0(M)$, and $\overline{\mathcal{F}}_t$ -adapted $h \in L^2(\Omega \rightarrow \mathbf{H}; \overline{\mathbb{P}})$,*

$$\mathbb{E}_{\overline{\mathbb{P}}} \langle DF(X), h + u_0^{-1} U(X_0) \rangle_{\mathbf{H}} = \mathbb{E}_{\overline{\mathbb{P}}} \left\{ F(X) \left(\int_0^T \langle h'(t), dB_t \rangle - (\text{div}_\mu^0 U)(X_0) \right) \right\}$$

One can mimic the proof of [12], we omit it here. Due to this lemma, we have the following result, the main idea is standard.

Theorem 4.5. *Assume (A). $(\mathcal{E}^\mu, \mathcal{FC}_0^\infty)$ is closable in $L^2(W^T; \Pi_\mu^T)$, and its closure is symmetric Dirichlet form.*

Proof. It suffices to prove the closability. Let $\{F_n\}_{n \geq 1} \subset \mathcal{FC}_0^\infty$ such that $\lim_{n \rightarrow \infty} F_n = 0$ in $L^2(W^T; \Pi_\mu^T)$ and $V := \lim_{n \rightarrow \infty} DF_n(X)$ exists in $L^2(\overline{\Omega} \rightarrow \mathbf{H}; \overline{\mathbb{P}})$. We intend to prove that $V = 0$.

Under the condition **(A)**, by the decomposition of identity, there is a smooth ONB $\{U_i\}_{i=1}^d$ for the tangent space. Then for any $f \in C_0^\infty(M^n)$, define

$$\xi_j = \sum_{i=1}^d \langle u_0 Q_{t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, X_{t_2}, \dots, X_{t_n}), U_j(X_0) \rangle_0, \quad j = 1, 2, \dots, d,$$

and there holds

$$\sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, X_{t_2}, \dots, X_{t_n}) = \sum_{i=1}^\infty \xi_i u_0^{-1} U_i(X_0). \quad (4.3)$$

Combining this with (4.1), it suffices to prove $\mathbb{E}_{\overline{\mathbb{P}}} \langle V, h + \xi u_0^{-1} U(X_0) \rangle_{\mathbf{H}} = 0$ for $\overline{\mathcal{F}}_t$ -adapted $h \in L^2(\overline{\Omega} \rightarrow \mathbf{H}_0; \overline{\mathbb{P}})$, $\xi \in L^2(\overline{\Omega}; \overline{\mathbb{P}})$ and $U \in \mathcal{H}_0(M)$. Since $\mathcal{F}C_0^\infty$ is dense in $L^2(W^T; \Pi_\mu^T)$, we may assume that $\xi = G(X)$ for some $G \in \mathcal{F}C_0^\infty$. In this case, it follows from Lemma 4.4 and Eq. (4.2) that

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}} \langle V, h + \xi u_0^{-1} U(X_0) \rangle_{\mathbf{H}} &= \mathbb{E}_{\overline{\mathbb{P}}} \langle V(0), \xi u_0^{-1} U(X_0) \rangle_{\mathbf{H}} = \lim_{n \rightarrow \infty} \mathbb{E}_{\overline{\mathbb{P}}} \langle DF_n(X), G(X) u_0^{-1} U(X_0) \rangle_{\mathbf{H}} \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_{\overline{\mathbb{P}}} \{ \langle \{D(F_n G)(X)\}(0), u_0^{-1} U(X_0) \rangle_{\mathbf{H}} - F_n(X) \langle ((DG)(X))(0), u_0^{-1} U(X_0) \rangle_{\mathbf{H}} \} \\ &= - \lim_{n \rightarrow \infty} \mathbb{E}_{\overline{\mathbb{P}}} \{ F_n(X) (G(X) (\operatorname{div}_\mu^0 U)(X_0) + \langle ((DG)(X))(0), u_0^{-1} U(X_0) \rangle_{\mathbf{H}}) \} \\ &= 0. \end{aligned}$$

□

By Theorems 4.3 and 4.5, we obtain the main result of this subsection as follows.

Theorem 4.6. *Assume **(A)**. If the log-Sobolev inequality*

$$\mu(f^2 \log f^2) \leq C \mu(|\nabla^0 f|_0^2), \quad f \in C_b^1(M), \quad \mu(f^2) = 1 \quad (4.4)$$

holds for some constant $C > 0$, then

$$\Pi_\mu^T(F^2 \log F^2) \leq (2 \vee C) \mathcal{E}^\mu(F, F), \quad F \in \mathcal{D}(\mathcal{E}^\mu), \quad \Pi_\mu^T(F^2) = 1.$$

Proof. From Theorem 4.5, it suffice to prove $F \in \mathcal{F}C_0^\infty$. By Theorem 4.3 and the condition (4.4), we have

$$\begin{aligned} \Pi_\mu^T(F^2 \log F^2) &= \int_M \Pi_x^T(F^2 \log F^2) \mu(dx) \\ &\leq 2 \int_M \mathcal{E}^x(F, F) \mu(dx) + C \int_M \Pi_x^T(F^2) \log \Pi_x^T(F^2) \mu(dx) \\ &\leq 2 \mathbb{E}_{\mathbb{P}} \|D^0 F(X)\|_{\mathbf{H}_0}^2 + C \int_M |\nabla^0 \sqrt{\mathbb{E} F^2(X)}|_0^2 d\mu. \end{aligned} \quad (4.5)$$

Moreover, letting $F(X) = f(X_{t_1}, X_{t_2}, \dots, X_{t_n})$, it follows from Lemma 3.3 that

$$\begin{aligned} |\nabla^0 \sqrt{\mathbb{E} F^2(X)}|_0^2 &= \frac{|\mathbb{E} F(X) \sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \dots, X_{t_n})|^2}{\mathbb{E} F^2(X)} \\ &\leq \mathbb{E} \left| \sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \dots, X_{t_n}) \right|^2. \end{aligned} \quad (4.6)$$

Combining (4.6) with (4.5), we complete the proof. □

5 Transportation-cost inequalities on path spaces over convex manifolds

The main purpose of this section is to investigate the Talagrand type inequalities on the path space $W^{S,T} := C([S,T]; M)$, $0 \leq S < T < T_c$ of the (reflecting) diffusion processes on the manifold with convex boundary under g_t , $t \in [0, T_c]$. Let X_t be the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by L_t with initial distribution $\mu \in \mathcal{P}(M)$. Assume that X_t is non-explosive, which is the case if

$$\mathcal{R}_t^Z \geq K(t), \text{ for some } K \in C([0, T_c]), \mathbb{I}_t \geq 0. \quad (5.1)$$

We call the metric flow is convex flow if $(\partial M, g_t)$ keeps convex, i.e. $\mathbb{I}_t \geq 0$. Let $\Pi_\mu^{S,T}$ be the distribution of $X_{[S,T]} := \{X_t : t \in [S, T]\}$, $0 \leq S < T < T_c$, which is a probability measure on the (free) path space $W^{S,T}$. When $\mu = \delta_x$, we denote $\Pi_{\delta_x}^{S,T} = \Pi_x^{S,T}$. For any nonnegative measurable function F on $W^{S,T}$ such that $\Pi_\mu^{S,T}(F) = 1$, one has

$$\mu_F^{S,T}(\mathrm{d}x) := \Pi_x^{S,T}(F)\mu(\mathrm{d}x) \in \mathcal{P}(M). \quad (5.2)$$

Consider the uniform distance on $W^{S,T}$:

$$\rho_\infty(\gamma, \eta) := \sup_{t \in [S, T]} \rho(\gamma_t, \eta_t), \gamma, \eta \in W^{S,T}.$$

Let $W_2^{\rho_\infty}$ be the L^2 -Wasserstein distance (or L^2 -transportation cost) induced by ρ_∞ . In general, for any $p \in [1, \infty)$ and for two probability measures Π_1, Π_2 on $W^{S,T}$,

$$W_p^{\rho_\infty}(\Pi_1, \Pi_2) := \inf_{\pi \in \mathcal{C}(\Pi_1, \Pi_2)} \left\{ \int_{W^{S,T} \times W^{S,T}} \rho_\infty(\gamma, \eta)^p \pi(\mathrm{d}\gamma, \mathrm{d}\eta) \right\}^{1/p}$$

is the L^p -Wasserstein distance (or L^p -transportation cost) of Π_1 and Π_2 , induced by the uniform norm, where $\mathcal{C}(\Pi_1, \Pi_2)$ is the set of all couplings for Π_1 and Π_2 . Similarly the L^p -Wasserstein distance of μ and ν induced by g_t -distance defined by

$$W_{p,t}(\nu, \mu) = \inf_{\eta \in \mathcal{C}(\nu, \mu)} \left\{ \int_{M \times M} \rho_t(x, y)^p \mathrm{d}\eta(x, y) \right\}^{1/p}.$$

The following Theorem 5.2 provides that there are some transportation-cost inequalities to be equivalent to the lower bound of \mathcal{R}_t^Z and the convexity of $(\partial M, g_t)$, $t \in [0, T_c]$ (when $\partial M \neq \emptyset$). To prove this result, we need the following lemma due to [?].

Lemma 5.1. *Let μ be a probability measure on M and $f \in C_b^2(M)$ such that $\mu(f) = 0$. For small enough $\varepsilon > 0$ such that $f_\varepsilon := 1 + \varepsilon f \geq 0$, there holds*

$$\mu(f^2) \leq \frac{1}{\varepsilon} \sqrt{\mu(|\nabla^s f|_s^2)} W_{2,s}(f_\varepsilon \mu, \mu) + \frac{\|\text{Hess}_f^s\|_\infty}{2\varepsilon} W_{2,s}(f_\varepsilon \mu, \mu)^2,$$

where $\|\text{Hess}_f^s\|_\infty = \sup_{x \in M} \|\text{Hess}_f^s\|_{op}$ for $\|\cdot\|_{op}$ the operator norm in \mathbb{R}^d .

The main result of the section is presented as follows.

Theorem 5.2. Let $P_{S,T}(x, \cdot)$ be the distribution of X_T with conditional $X_S = x$. Denote the corresponding inhomogeneous semigroup by $\{P_{S,T}\}_{0 \leq S \leq T < T_c}$. For any $p \in [1, \infty)$, the following statements are equivalent to each other:

(1) (5.1) holds.

(2) For any $0 \leq S \leq T < T_c$, $\mu \in \mathcal{P}(M)$ and nonnegative F with $\Pi_\mu^{S,T}(F) = 1$,

$$W_2^{\rho_\infty}(F\Pi_\mu^{S,T}, \Pi_{\mu_F}^{S,T}) \leq 4C(S, T, K)\Pi_\mu^{S,T}(F \log F)$$

holds, where $\mu_F^{S,T} \in \mathcal{P}(M)$ is fixed by (5.2) and $C(S, T, K) := \sup_{t \in [S, T]} \int_S^t e^{-2 \int_u^t K(r) dr} du$, which will keep the same meaning in (3), (7) and (8).

(3) For any $x \in M$ and $0 \leq S \leq T < T_c$,

$$W_2^{\rho_\infty}(F\Pi_x^{S,T}, \Pi_x^{S,T})^2 \leq 4C(S, T, K)\Pi_x^{S,T}(F \log F), \quad F \geq 0, \quad \Pi_x^T(F) = 1.$$

(4) For any $x \in M$ and $0 \leq S \leq T < T_c$,

$$W_{2,T}(P_{S,T}(x, \cdot), fP_{S,T}(x, \cdot))^2 \leq 4 \left(\int_S^T e^{-2 \int_u^T K(r) dr} du \right) P_{S,T}(f \log f)(x), \quad f \geq 0, \quad P_{S,T}f(x) = 1.$$

(5) For any $x \in M$ and $0 \leq S \leq T < T_c$,

$$W_{2,T}(P_{S,T}(x, \cdot), fP_{S,T}(x, \cdot))^2 \leq 4 \left(\int_S^T e^{-2 \int_u^T K(r) dr} du \right)^2 P_{S,T} \frac{|\nabla^T f|_T^2}{f}(x),$$

where $f \geq 1$ and $P_{S,T}f = 1$.

(6) For $0 \leq S \leq T < T_c$ and $\mu, \nu \in \mathcal{P}(M)$,

$$W_p^{\rho_\infty}(\Pi_\mu^{S,T}, \Pi_\nu^{S,T}) \leq e^{-\int_S^T K(r) dr} W_{p,S}(\mu, \nu).$$

(7) For any $0 \leq S \leq T < T_c$, $\mu \in \mathcal{P}(M)$, and $F \geq 0$ with $\Pi_\mu^{S,T}(F) = 1$,

$$W_2^{\rho_\infty}(F\Pi_\mu^{S,T}, \Pi_\mu^{S,T}) \leq 2 \{C(S, T, K)\Pi_\mu^{S,T}(F \log F)\}^{1/2} + e^{-\int_S^T K(r) dr} W_{2,S}(\mu_F^{S,T}, \mu).$$

(8) For any $\mu \in \mathcal{P}(M)$ and $C \geq 0$ such that

$$W_{2,S}(f\mu, \mu)^2 \leq C\mu(f \log f), \quad f \geq 0, \quad \mu(f) = 1,$$

there holds

$$W_2^{\rho_\infty}(F\Pi_\mu^{S,T}, \Pi_\mu^{S,T}) \leq \left(2\sqrt{C(S, T, K)} + \sqrt{C}e^{-\int_S^T K(r) dr} \right)^2 \Pi_\mu^{S,T}(F \log F),$$

for $F \geq 0$, $\Pi_\mu^{S,T}(F) = 1$.

Proof. By taking $\mu = \delta_x$, we have $\mu_F^T = \Pi_x^T(F)\delta_x = \delta_x$. It is easy to see that (3) follows from (2), (7) and (8). (4) follows from (3) by taking $F(X_{[S,T]}) = f(X_T)$. (6) implies

$$W_{p,S}(\delta_x P_{S,T}, \delta_y P_{S,T}) \leq e^{-\int_S^T K(r)dr} \rho_S(x, y)$$

and thus implies (1) by [7, Theorem 5.3]. Moreover, it is clear that (8) follows from (7) while (7) is implied by each of (2) and (6). “(3) \Rightarrow (2)” is the same as explained in time-homogeneous case (see [24, the proof of Theorem 1.1 (b)]). So it suffices to prove (1) \Rightarrow (3), each of (4) and (5) \Rightarrow (1), (1) \Rightarrow (5), (1) \Rightarrow (6). Without loss generality, we assume $S = 0$ for simplicity.

(a) **(1) implies (3)** We shall only consider the case where ∂M is non-empty. For the case without boundary, the following argument works well by taking $l_t = 0$ and $N_t = 0$. Simply denote $X_{[0,T]}^x = X_{[0,T]}$. Let F be a positive bounded measurable function on W^T such that $\inf F > 0$ and $\Pi_x^T(F) = 1$. Let $d\mathbb{Q} = F(X_{[0,T]})d\mathbb{P}$. Since $\mathbb{E}F(X_{[0,T]}) = \Pi_\mu^T(F) = 1$, \mathbb{Q} is a probability measure on Ω . Then, with a similar discussion as in [25], we conclude that there exists a unique \mathcal{F}_t -predict process β_t on \mathbb{R}^d such that

$$F(X_{[0,T]}) = m_T = e^{\int_0^T \langle \beta_s, dB_s \rangle - \frac{1}{2} \int_0^T \|\beta_s\|^2 ds}$$

and

$$\int_0^T \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 ds = 2\mathbb{E}F(X_{[0,T]}) \log X_{[0,T]}. \quad (5.3)$$

Then by the Girsanov theorem, $\tilde{B}_t := B_t - \int_0^t \beta_s ds$, $t \in [0, T]$ is a d -dimensional Brownian motion under the probability measure \mathbb{Q} .

Let Y_t solve the following SDE

$$dY_t = \sqrt{2}P_{X_t, Y_t}^t u_t \circ d\tilde{B}_t + Z_t(Y_t)dt + N_t(Y_t)d\tilde{l}_t, \quad Y_0 = x, \quad (5.4)$$

where P_{X_t, Y_t}^t is the g_t -parallel displacement along the minimal geodesic from X_t to Y_t and \tilde{l}_t is the local time of Y_t on ∂M . As announced, under \mathbb{Q} , \tilde{B}_t is a d -dimensional Brownian motion, the distribution of $Y_{[0,T]}$ is Π_x^T .

On the other hand, since $\tilde{B}_t = B_t - \int_0^t \beta_s ds$, we have

$$dX_t = \sqrt{2}u_t \circ d\tilde{B}_t + Z_t(X_t)dt + \sqrt{2}u_t \beta_t dt + N_t(X_t)d\tilde{l}_t. \quad (5.5)$$

Moreover, for any bounded measurable function G on W^T ,

$$\mathbb{E}_{\mathbb{Q}}G(X_{[0,T]}) := \mathbb{E}(FG)(X_{[0,T]}) = \Pi_x^T(FG).$$

We conclude that the distribution of $X_{[0,T]}$ under \mathbb{Q} coincides with $F\Pi_x^T$. Therefore,

$$W_2^{\rho_\infty}(F\Pi_x^T, \Pi_x^T)^2 \leq \mathbb{E}_{\mathbb{Q}}\rho_\infty(X_{[0,T]}, Y_{[0,T]})^2 = \mathbb{E}_{\mathbb{Q}} \max_{t \in [0, T]} \rho_t(X_t, Y_t)^2. \quad (5.6)$$

By the convexity of $(\partial M, g_t)$, we have

$$\langle N_t(x), \nabla^t \rho_t(\cdot, y)(x) \rangle_t = \langle N_t(x), \nabla^t \rho_t(y, \cdot)(x) \rangle_t \leq 0.$$

Combining this with (5.4) and (5.5), and by the Itô formula, we obtain from $\mathcal{R}_t^Z \geq K(t)$ that

$$\begin{aligned} d\rho_t(X_t, Y_t) &\leq -K(t)\rho_t(X_t, Y_t)dt + \sqrt{2}\langle u_t\beta_t, \nabla^t \rho_t(\cdot, Y_t)(X_t) \rangle_t dt \\ &\leq (-K(t)\rho_t(X_t, Y_t) + \sqrt{2}\|\beta_t\|)dt. \end{aligned} \quad (5.7)$$

Since $X_0 = Y_0 = x$, this implies

$$\begin{aligned} \rho_t(X_t, Y_t)^2 &\leq e^{-2\int_0^t K(r)dr} \left(\sqrt{2} \int_0^t e^{\int_0^s K(r)dr} \|\beta_s\| ds \right)^2 \\ &\leq 2e^{-2\int_0^t K(r)dr} \int_0^t e^{2\int_0^s K(r)dr} ds \cdot \int_0^t \|\beta_s\|^2 ds, \quad t \in [0, T]. \end{aligned} \quad (5.8)$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}} \max_{t \in [0, T]} \rho_t(X_t, Y_t)^2 \leq 2 \max_{t \in [0, T]} \int_0^t e^{-2\int_s^t K(r)dr} ds \int_0^t \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 ds. \quad (5.9)$$

Therefore, (3) follows from (5.6) and (5.3).

(b) **(4) implies (1)** Let $f \in C_b^2(M)$ such that $P_{0,T}f(x) = 0$. Then, for small $\varepsilon > 0$ such that $f_\varepsilon := 1 + \varepsilon f \geq 0$, we have

$$P_{0,T}(f_\varepsilon \log f_\varepsilon) = P_{0,T} \left\{ (1 + \varepsilon f) \left(\varepsilon f - \frac{1}{2}(\varepsilon f)^2 + o(\varepsilon^2) \right) \right\} (x) = \frac{\varepsilon^2}{2} P_{0,T}f^2(x) + o(\varepsilon^2).$$

Combining with lemma 5.1 and (4), we obtain

$$\begin{aligned} (P_{0,T}f^2)^2(x) &\leq 4 \int_0^T e^{-2\int_u^T K(r)dr} du \cdot P_{0,T}|\nabla^T f|_T^2(x) \cdot \lim_{\varepsilon \rightarrow 0} \frac{P_{0,T}f_\varepsilon \log f_\varepsilon(x)}{\varepsilon^2} \\ &\quad 4 \int_0^T e^{-2\int_u^T K(r)dr} du \cdot P_{0,T}|\nabla^T f|_T^2(x) P_{0,T}f^2(x). \end{aligned}$$

This is equivalent to [7, Theorem 5.3] for $\sigma = 0, p = 2$ and continuous function K . Therefore, by [7, Theorem 5.3] “(2) \Leftrightarrow (1)”, (4) implies (1).

(c) **(5) is equivalent to (1)**. Similarly to (b), combining condition (5) with Lemma 5.1, we obtain

$$\begin{aligned} P_{0,T}f^2(x) &\leq 2 \int_0^T e^{-2\int_s^T K(r)dr} ds \sqrt{P_{0,T}|\nabla^T f|_T^2(x)} \lim_{\varepsilon \rightarrow 0} \sqrt{P_{0,T} \frac{|\nabla^T f_\varepsilon|_T^2}{f_\varepsilon \varepsilon^2}(x)} \\ &= 2 \int_0^T e^{-2\int_s^T K(r)dr} ds P_{0,T}|\nabla^T f|_T^2. \end{aligned}$$

Hence, (1) holds.

On the other hand, due to (1) \Rightarrow (4) and (5.1),

$$P_{0,T}(f \log f)(x) \leq \int_0^T e^{-2\int_s^T K(r)dr} ds \cdot P_{0,T} \frac{|\nabla^T f|_T^2}{f}(x), \quad f \geq 0, \quad P_{0,T}f(x) = 1,$$

we conclude that (1) implies (5).

(e) **(1) implies (2)**. For any $x, y \in M$, there exists $\Pi_{x,y} \in \mathcal{C}(\Pi_x^T, \Pi_y^T)$ such that

$$\int_{W^T \times W^T} \rho_\infty^p d\Pi_{x,y} \leq e^{-p\int_0^T K(r)dr} \rho_0(x, y)^p.$$

The following discussion is similar with the constant matrix case (see [24, Theorem 1.1]). \square

6 Extension to non-convex setting

In this section, we first consider $L_t = \psi_t^2(\nabla^t + Z_t)$ with diffusion coefficient ψ_t on manifolds with convex flow; then extend these results to non-convex case.

6.1 The case with a diffusion coefficient

Let $\psi_t(\cdot) = \psi(t, \cdot) > 0$ be a smooth function on (M, g_t) and $\Pi_{\mu, \psi}^T$ be the distribution of the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by $L_t = \psi_t^2(\nabla^t + Z_t)$ on time interval $[0, T] \subset [0, T_c)$ with initial distribution μ . Set $\Pi_{x, \psi}^T = \Pi_{\delta_x, \psi}^T$ for $x \in M$. Moreover, for $F \geq 0$ with $\Pi_{\mu, \psi}^T = 0$, let $\mu_{F, \psi}^T(dx) = \Pi_{s, \psi}^T(F)\mu(dx)$.

Theorem 6.1. *Assume that $\mathbb{I}_t \geq 0$ for $t \in [0, T_c)$ and $\text{Ric}_t^Z \geq K_1(t)$, $\mathcal{G}_t \leq K_2(t)$ for some continuous function on $[0, T_c)$. Let $\psi \in C_b^{1, \infty}([0, T_c) \times M)$ be strictly positive. Let*

$$K_\psi(t) = (d-1)\|\nabla^t \psi_t\|_\infty^2 + K_1^-(t)\|\psi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t \psi_t\|_\infty + K_2(t).$$

Then

$$W_2^{\rho\infty}(F\Pi_{\mu, \psi}^T, \Pi_{\mu_{F, \psi}^T, \psi}^T)^2 \leq C(T, \psi)\Pi_{\mu, \psi}^T(F \log F), \quad \mu \in \mathcal{P}(M), \quad F \geq 0, \quad \Pi_{\mu, \psi}^T(F) = 1$$

holds for

$$C(T, \psi) := \inf_{R>0} \left\{ 4(1+R^{-1}) \int_0^T \|\psi_s\|_\infty^2 e^{2 \int_s^T K_\psi(r) dr} ds \cdot \exp \left[8(1+R) \sup_{s \in [0, T]} \|\nabla^s \psi_s\|_\infty \right] \right\}.$$

Proof. We shall only consider the case that ∂M is non-empty. As explained in the proof of “(3) \Rightarrow (2)” in [24, Theorem 4.1], it suffices to prove for $\mu = \delta_x$, $x \in M$. In this case, the desired inequality reduces to

$$W_2^{\rho\infty}(F\Pi_{x, \psi}^T, \Pi_{x, \psi}^T) \leq 2C(T, \psi)\Pi_{x, \psi}^T(F \log F), \quad F \geq 0, \quad \Pi_{x, \psi}^T(F) = 1.$$

Since the diffusion coefficient is non-constant, it is convenient to adopt the Itô differential d_I for the Girsanov transformation. So the L_t -reflecting diffusion process can be constructed by solving the Itô SDE

$$d_I X_t = \sqrt{2}\psi_t(X_t)u_t dB_t + \psi_t^2(X_t)Z_t(X_t)dt + N_t(X_t)dl_t, \quad X_0 = x,$$

where B_t is the d -dimensional Brownian motion with natural filtration \mathcal{F}_t . Let β_t , \mathbb{Q} and \tilde{B}_t be the same as in the proof of Theorem 5.2. Then

$$d_I X_t = \sqrt{2}\psi_t(X_t)u_t d\tilde{B}_t + \{\psi_t^2(X_t)Z_t(X_t) + \sqrt{2}\psi_t(X_t)u_t \beta_t\}dt + N_t(X_t)dl_t. \quad (6.1)$$

Let Y_t solve

$$d_I Y_t = \sqrt{2}\psi_t(Y_t)P_{X_t, Y_t}^t u_t d\tilde{B}_t + \psi_t^2(Y_t)Z_t(Y_t)dt + N_t(Y_t)d\tilde{l}_t, \quad Y_0 = y, \quad (6.2)$$

where \tilde{l}_t is the local time of Y_t on ∂M . As explained in the above theorem (see e.g. the proof of Theorem 5.2 (a)), under \mathbb{Q} , the distribution of $Y_{[0,T]}$ and $X_{[0,T]}$ are $\Pi_{x,\psi}^T$ and $F\Pi_{x,\psi}^T$, so

$$W_2^{\rho\infty}(F\Pi_{x,\psi}^T, \Pi_{x,\psi}^T) \leq \mathbb{E}_{\mathbb{Q}} \max_{t \in [0,T]} \rho_t(X_t, Y_t)^2. \quad (6.3)$$

Noting that due to the convexity of the boundary,

$$\langle N_t(x), \nabla^t \rho(\cdot, y)(x) \rangle_t = \langle N_t(y), \nabla^t \rho(y, \cdot)(x) \rangle_t \leq 0, \quad x \in \partial M,$$

Combining this with (6.1), (6.2) and the comparison theorem [7, Theorem 4.1], we obtain

$$\begin{aligned} d\rho_t(X_t, Y_t) &\leq \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t)) \langle \nabla^t \rho(\cdot, Y_t)(X_t), u_t dB_t \rangle_t \\ &\quad + K_\psi(t) \rho_t(X_t, Y_t) dt + \sqrt{2} \|\psi_t\|_\infty \|\beta_t\| dt, \end{aligned}$$

where

$$K_\psi(t) = (d-1) \|\nabla^t \psi_t\|_\infty^2 + K_1^-(t) \|\psi_t\|_\infty^2 + 2 \|Z_t\|_\infty \|\nabla^t \psi_t\|_\infty \|\psi_t\|_\infty + K_2(t).$$

Then

$$M_t := \sqrt{2} \int_0^t e^{-\int_0^s K_\psi(r) dr} (\psi_s(X_s) - \psi_s(Y_s)) \langle \nabla^s \rho_s(\cdot, Y_s)(X_s), u_s d\tilde{B}_s \rangle_s$$

is a \mathbb{Q} -martingale such that

$$\rho_t(X_t, Y_t) \leq e^{\int_0^t K_\psi(r) dr} \left(M_t + \sqrt{2} \int_0^t e^{-\int_0^s K_\psi(r) dr} \|\psi_s\|_\infty \|\beta_s\| ds \right), \quad t \in [0, T].$$

So by the Doob inequality, we obtain

$$\begin{aligned} h_t &:= e^{-2 \int_0^t K_\psi(s) ds} \mathbb{E} \max_{s \in [0,t]} \rho_s(X_s, Y_s)^2 \\ &\leq (1+R) \mathbb{E}_{\mathbb{Q}} \max_{s \in [0,t]} M_s^2 + 2(1+R^{-1}) \mathbb{E}_{\mathbb{Q}} \left(\int_0^t e^{-\int_0^s K_\psi(r) dr} \|\psi_s\|_\infty \|\beta_s\| ds \right)^2 \\ &\leq 4(1+R) \mathbb{E}_{\mathbb{Q}} M_t^2 + 2(1+R^{-1}) \int_0^t e^{-2 \int_0^s K_\psi(r) dr} \|\psi_s\|_\infty^2 ds \int_0^t \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 ds \\ &\leq 8(1+R) \sup_{s \in [0,T]} \|\nabla^s \psi_s\|_\infty \int_0^t h_s ds \\ &\quad + 2(1+R^{-1}) \int_0^T \|\psi_s\|_\infty^2 e^{-2 \int_0^s K_\psi(r) dr} ds \cdot \int_0^T \mathbb{E} \|\beta_s\|^2 ds, \quad t \in [0, T]. \end{aligned}$$

Since $h_0 = 0$, this inequality implies

$$\begin{aligned} e^{-2 \int_0^T K_\psi(s) ds} \mathbb{E}_{\mathbb{Q}} \max_{s \in [0,T]} \rho_s(X_s, Y_s)^2 &= h_T \\ &\leq 2(1+R^{-1}) \int_0^T \|\psi_s\|_\infty^2 e^{-2 \int_0^s K_\psi(r) dr} ds \cdot \exp \left[8(1+R) \sup_{s \in [0,T]} \|\nabla^s \psi_s\|_\infty \right] \cdot \int_0^T \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 ds. \end{aligned}$$

As explained in (5.3), it holds

$$\int_0^T \mathbb{E}_{\mathbb{Q}} \|\beta_s\|^2 ds = 2 \mathbb{E} F(X_{[0,T]}) \log F(X_{[0,T]}).$$

Therefore

$$\begin{aligned} & \mathbb{E} \max_{s \in [0, T]} \rho_s(X_s, Y_s)^2 \\ & \leq 4(1 + R^{-1}) \int_0^T \|\psi_s\|_\infty^2 e^{2 \int_s^T K_\psi(r) dr} ds \cdot \exp \left[8(1 + R) \sup_{s \in [0, T]} \|\nabla^s \psi_s\|_\infty \right] \Pi_{o, \psi}^T(F \log F). \end{aligned}$$

Combining with (6.3), we complete the proof. \square

Theorem 6.2. *In the situation of Theorem 6.1,*

$$W_2^{\rho_\infty}(\Pi_{\nu, \psi}^T, \Pi_{\mu, \psi}^T) \leq 2e^{\int_0^T (K_\psi(t) + \|\nabla^t \psi_t\|_\infty) dt} W_{2,0}(\nu, \mu).$$

Proof. As explained in the proof of Theorem 5.2 “(6) \Rightarrow (5)”, we only consider $\nu = \delta_x$, and $\mu = \delta_y$. Let X_t and Y_t solve the following SDEs respectively.

$$\begin{aligned} d_I X_t &= \sqrt{2} \psi_t(X_t) u_t dB_t + \psi_t^2(X_t) Z_t(X_t) dt + N_t(X_t) d\tilde{l}_t, \quad X_0 = x; \\ d_I Y_t &= \sqrt{2} \psi_t(Y_t) P_{X_t, Y_t}^t u_t dB_t + \psi_t^2(Y_t) Z_t(Y_t) dt + N_t(Y_t) d\tilde{l}_t, \quad Y_0 = y. \end{aligned}$$

Then, as explained in Theorem 6.1, by the Itô formula,

$$d\rho_t(X_t, Y_t) \leq \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t)) \langle \nabla^t \rho_t(\cdot, Y_t)(X_t), u_t dB_t \rangle_t + K_\psi(t) \rho_t(X_t, Y_t) dt. \quad (6.4)$$

Therefore,

$$\rho_t(X_t, Y_t) \leq e^{\int_0^t K_\psi(s) ds} (M_t + \rho_0(x, y)), \quad t \geq 0, \quad (6.5)$$

for $M_t := \sqrt{2} \int_0^t e^{-\int_0^s K_\psi(u) du} (\psi_s(X_s) - \psi_s(Y_s)) \langle \nabla^s \rho_s(\cdot, Y_s)(X_s), u_s dB_s \rangle_s$. Again using the Itô formula,

$$d\rho_t^2(X_t, Y_t) \leq d\tilde{M}_t + 2 [K_\psi(t) + \|\nabla^t \psi_t\|_\infty^2] \rho_t(X_t, Y_t)^2 dt,$$

where $d\tilde{M}_t = 2\rho_t(X_t, Y_t)(\psi_t(X_t) - \psi_t(Y_t)) \langle \nabla^t \rho_t(\cdot, Y_t)(X_t), u_t dB_t \rangle$. This implies

$$d\rho_t^2(X_t, Y_t) \leq e^{2 \int_0^t (K_\psi(s) + \|\nabla^s \psi_s\|_\infty) ds} \rho_0(x, y)^2.$$

Combining this with (6.5), we arrive at

$$\begin{aligned} W_2^{\rho_\infty}(\Pi_{x, \psi}^T, \Pi_{y, \psi}^T)^2 & \leq \mathbb{E} \max_{t \in [0, T]} \rho_t(X_t, Y_t)^2 \leq e^{2 \int_0^T K_\psi(t) dt} \mathbb{E} \max_{t \in [0, T]} (M_t + \rho_0(x, y))^2 \\ & \leq 4e^{2 \int_0^T K_\psi(t) dt} \mathbb{E} (M_T + \rho_0(x, y))^2 = 4e^{2 \int_0^T K_\psi(t) dt} \mathbb{E} (M_T^2 + \rho_0^2(x, y)) \\ & \leq 4e^{2 \int_0^T K_\psi(t) dt} \left(\rho_0^2(x, y) + 2 \int_0^T e^{-2 \int_0^s K_\psi(u) du} \|\nabla^s \psi_s\|_\infty \mathbb{E} \rho_s(X_s, Y_s)^2 ds \right) \\ & \leq 4e^{2 \int_0^T (K_\psi(t) + \|\nabla^t \psi_t\|_\infty) dt} \rho_0^2(x, y) \end{aligned}$$

where the second inequality is due to the Doob inequality. This implies the desired inequality for $\mu = \delta_x$ and $\nu = \delta_y$. \square

6.2 Non-convex manifold

As discussed in Theorem 6.1 and with a proper conformal change of metric, we are able to establish the following transportation-cost inequality on a class of manifolds with non-convex boundary. Let

$$\mathcal{D} = \{\phi \in C_b^2([0, T_c) \times M) : \inf \phi_t = 1, \mathbb{I}_t \geq -N_t \log \phi_t\}.$$

Assume that $\mathcal{D} \neq \emptyset$ and for some $K_1, K_2 \in C([0, T_c))$ such that

$$\text{Ric}_t^Z \geq K_1(t), \quad \mathcal{G}_t \leq K_2(t) \quad (6.6)$$

holds. To make the boundary convex, let $\phi_t \in \mathcal{D}$. By Theorem [23, Lemma 2.1], ∂M become convex under $\tilde{g}_t = \phi_t^{-2} g_t$. Let $\tilde{\Delta}_t$ and $\tilde{\nabla}^t$ be the Laplacian and gradient induced by the new metric \tilde{g}_t . Since $\phi_t \geq 1$, $\rho_t(x, y)$ is large than $\tilde{\rho}_t(x, y)$, the Riemannian \tilde{g}_t -distance between x and y .

Theorem 6.3. *Let $\partial M \neq \emptyset$ and $\mathbb{I}_t \geq -\sigma(t)$ for positive $\sigma \in C([0, T_c))$. Assume (6.6) holds. For $\phi \in \mathcal{D}$, let*

$$K_\phi(t) := (d-1)\|\nabla^t \phi_t\|_\infty^2 + K_{\phi,1}^-(t) + 2\|\phi_t Z_t + (d-2)\nabla^t \phi_t\|_\infty \|\nabla^t \phi_t\|_\infty + K_{\phi,2}(t),$$

where

$$\begin{aligned} K_{\phi,1}(t) &:= \inf\{\phi_t K_1(t) + \frac{1}{2}L_t \phi_t^2 - |\nabla^t \phi_t^2|_t |Z_t|_t - (d-2)|\nabla^t \phi_t|_t^2\}, \\ K_{\phi,2}(t) &:= \sup\{-2\partial_t \log \phi_t + K_2(t)\}. \end{aligned}$$

Then for any $\mu \in \mathcal{P}(M)$,

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F}^T) \leq \sup_{s \in [0, T]} \|\phi_t\|_\infty^2 C(T, \phi) \Pi_\mu^T(F \log F), \quad F \geq 0, \quad \Pi_\mu^T(F) = 1$$

holds for

$$C(T, \phi) = \inf_{R > 0} \left\{ 4(1 + R^{-1}) \int_0^T e^{2 \int_s^T K_\phi(r) dr} ds \exp \left[8(1 + R) \sup_{s \in [0, T]} \|\nabla^s \phi_s\|_\infty \right] \right\}.$$

Proof. According to the proof of [7, Proposition 4.7]. We have $L_t := \phi_t^{-2}(\tilde{\Delta}_t + \tilde{Z}_t)$, where $\tilde{Z}_t = \phi_t^2 Z_t + \frac{d-2}{2} \nabla^t \phi_t^2$, and

$$\widetilde{\text{Ric}}_t^{\tilde{Z}} \geq K_{\phi,1}(t), \quad \tilde{\mathcal{G}}_t \leq K_{\phi,2}(t).$$

Let K_ψ be defined in Theorem 6.3 for the manifold equipped with \tilde{g}_t . Then, $L_t = \psi_t^2(\tilde{\Delta}_t + \tilde{Z}_t)$, where $\psi_t = \phi_t^{-1}$, we see that $K_\psi(t) \leq K_\phi(t)$ and thus $C(t, \psi) \leq C(t, \phi)$. Hence, it follows from Theorem 6.3 that

$$W_2^{\tilde{\rho}_\infty}(F\Pi_\mu^T, \Pi_{\mu_F}^T)^2 \leq C(T, \phi) \Pi_\mu^T(F \log F), \quad F \geq 0, \quad \Pi_\mu^T(F) = 1,$$

where $\tilde{\rho}_\infty$ is the uniform distance on W^T induced by the metric \tilde{g}_t . The proof is completed by $\rho_\infty \leq \sup_{t \in [0, T]} \|\phi_t\|_\infty \tilde{\rho}_\infty$. \square

Since $K_\psi(t) \leq K_\phi(t)$ and $\tilde{\rho}_t \leq \rho_t \leq \|\phi_t\|_\infty \tilde{\rho}_t$, the following result follows from the proof of Theorem 3.2 by taking $\psi = \phi^{-1}$.

Theorem 6.4. *In the situation of Theorem 6.3,*

$$W_2^{\rho_\infty}(\Pi_\mu^T, \Pi_\nu^T) \leq 2 \sup_{t \in [0, T]} \|\phi_t\|_\infty e^{\int_0^T (K_\phi(t) + \|\nabla^t \phi_t\|_\infty) dt} W_{2,0}(\nu, \mu), \quad \mu, \nu \in \mathcal{P}, T > 0.$$

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